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A TIME-MODE APPROACH TO NONLINEAR VIBRATIONS OF ORTHOTROPIC THIN SHALLOW SPHERICAL SHELLS

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Abstract—A new analytical time-mode-assumption-based approach to nonlinear (large amplitude) axisymmetrical free vibrations of orthotropic thin shallow spherical shells is presented in this paper. The concept of the drift in theory of nonlinear dynamics is first adapted to modelling the asymmetry of vibration amplitudes of the shells. The nonlinearly-coupled algebraic-differential eigenvalue equations determining vibration frequencies and the drift are formulated with the variational method. Asymptotic solutions are obtained with a successive iteration method suggested by the author. The evolution of nonlinearities within a wider range of vibration amplitudes of the shell vibration with various parameters and boundary conditions is extensively investigated. It is generally concluded that very shallow spherical shells behave like flat circular plates when subject to free vibrations, while the vibration response for the moderately shallow spherical shells is complicated by the occurrence of both softening and hardening nonlinearities in different ranges of vibration amplitudes, and shallow spherical shells always tend to behave more like flat circular plates when subject to much larger amplitudes.

1. INTRODUCTION

Vibrations of shallow spherical shells with large amplitudes are nonlinear phenomena which have led to numerous studies for decades, since this topic is both of interest to engineers and of theoretical interest in that it is fundamental to nonlinearities of general physical systems. Singh *et al.* (1972), Ramachandran (1974, 1976) and Yasuda *et al.* (1984) investigated nonlinear axisymmetrical free or forced vibrations of isotropic shallow spherical shells respectively. More recently, Dumir (1986) first studied nonlinear axisymmetric free vibrations of orthotropic truncated thin shallow spherical and conical shells and achieved some significant results. However, investigations of this type of nonlinear problem always encounter the difficulty arising from nonlinearities in the governing equations modelling the physical problem. It is also observed that previous solutions of the problem were inevitably based upon space-mode assumptions. However, the space-mode itself is generally assumed based upon results of *static* deformation analysis, thus questioning its use for *dynamic* analysis. Most of the previous studies of this problem followed the Galerkin's method which was used to generate the governing equations describing the temporal modes, then the equation was solved using traditional numerical integration, perturbation or harmonic balance method etc. So far, there seems no improvement on the existing approaches to the problem.

Though now it is relatively simple to carry out a numerical analysis of the problem with the appearance of more and more advanced numerical techniques, an analytical solution, while being more difficult to achieve, does contain some important and fundamental physical insight into the problem itself. A reasonable formulation of the problem with some basic and physical consideration of the problem will always simplify the problem and render an analytical approach to the problem possible. Sinharay *et al.* (1985) has put forward a modified energy approach to the problem to simplify the analysis. His approach was still based upon space-mode assumptions, applying some experimental evidence to justify neglecting some coupling terms, thus decoupling the mathematical problem. His approach, though simpler, does not retain important coupling effects of the vibrations when amplitudes become relatively large.

This paper presents a completely new analytical approach to nonlinear vibrations of shallow shells of revolution. In particular, we investigate nonlinear axisymmetric free vibrations of orthotropic thin shallow spherical shells. Unlike all previous approaches, ours is put forward in the sense of time-mode assumptions based on the basic characteristics of vibrations of nonlinear systems. The deflection and stress function of the shells still assume the conventional mode-separable forms, i.e. temporal modes and spatial modes can be identified separately. The temporal modes take the form of harmonics while the spatial modes are totally unknown. To more accurately demonstrate the dynamics of the shells considered, we introduce a special quantity into the modal assumptions in order to model the asymmetric spring-like characteristics of the shells, which can be further recognized as a quantity related to the *drift* or *steady-streaming* of some nonlinear oscillators (Nayfeh and Mook, 1979). This quantity, like the vibration frequency, is an intrinsic parameter of the shells considered, and is related to the geometrical and material parameters, as well as the vibration amplitudes and boundary conditions of the shells.

The governing equations, which are formulated using the variational method, are eventually reduced to a set of nonlinearly-coupled algebraic-differential eigenvalue equations. These equations are then solved analytically with a successive iteration method that essentially stems from Yeh and Liu's *modified iteration method* dealing with nonlinear static stability of shallow shells (Yeh and Liu, 1965) and its counterpart suggested by the author (Li and Liu, 1991) that was utilized in studies of nonlinear vibrations of flat plates. Asymptotic expressions of the space-modes and an analytical relation for the amplitude-frequency response of the shells are finally derived. Extensive numerical results are presented for four usually-encountered boundary conditions. Effects of geometrical and orthotropy parameters on the evolution of nonlinear vibrations are investigated.

2. NONLINEAR EIGENVALUE PROBLEM FORMULATION

Consider an orthotropic thin shallow spherical shell subject to free axisymmetrical vibrations, as shown in Fig. 1. The von Kármán type governing equations of the present problem were given in the literature (Dumir, 1986) and elsewhere. To facilitate the subsequent analysis, we prefer to give the dynamic equilibrium equation in a variational form. This is routinely done by using Hamilton's principle. After writing out the Hamiltonian of the shell and applying integration by parts, we eventually arrive at a domain integral as

$$\int_{t_1}^{t_2}\int_0^a \left\{\gamma h \frac{\partial^2 w^*}{\partial t^2} + D\mathscr{L}_1(w^*) - \frac{1}{r}\frac{\partial}{\partial r}\left[\psi^*\left(\frac{\partial w^*}{\partial r} + \frac{2H^*}{a^2}r\right)\right]\right\}\delta w^* r\,\mathrm{d}r\,\mathrm{d}t = 0, \tag{1}$$

where δ indicates variational operations, $\mathscr{L}_1$ is a partial differential operator

$$\mathscr{L}_1(..) = \frac{\partial^4(..)}{\partial r^4} + \frac{2}{r}\frac{\partial^3(..)}{\partial r^3} - \frac{\beta}{r^2}\frac{\partial^2(..)}{\partial r^2} + \frac{\beta}{r^3}\frac{\partial(..)}{\partial r},$$

t is the time variable, a is the base radius of the shell, h is the thickness of the shell, H^* is the apex rise of the shell, γ is the mass density, w^* is the deflection, ψ^* is the stress function, β is the orthotropy parameter, D is the bending stiffness. In addition, we have the following relations (Dumir, 1986):

$$N_r = \frac{\psi^*}{r},\quad N_\theta = \frac{\partial\psi^*}{\partial r},\quad \beta = \frac{E_\theta}{E_r} = \frac{\nu_\theta}{\nu_r},\quad D = \frac{E_\theta h^3}{12(\beta - \nu_\theta^2)},$$

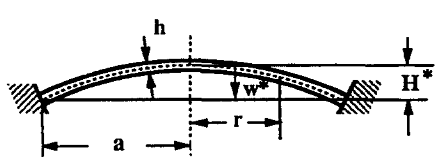

Fig. 1. Geometry and coordinates.

where E_r, E_θ are radial and circumferential Young's Moduli, ν_r, ν_θ are radial and circumferential Poisson's ratios, N_r and N_θ are radial and circumferential membrane stresses.

The accompanied natural boundary conditions may assume various forms. We mention here only four usually-encountered boundary conditions, which can be put into a compact form as follows:

$$r = 0: \quad w^* \text{ is finite}, \quad \frac{\partial w^*}{\partial r} = 0, \quad \lim_{r\to 0} r\left(\frac{\partial^3 w^*}{\partial r^3} + \frac{1}{r}\frac{\partial^2 w^*}{\partial r^2} - \frac{\beta}{r^2}\frac{\partial w^*}{\partial r}\right) = 0, \quad \psi^* = 0, \tag{2}$$

$$r = a: \quad w^* = 0, \quad \frac{\partial^2 w^*}{\partial r^2} + \frac{\lambda}{r}\frac{\partial w^*}{\partial r} = 0, \quad \frac{\partial \psi^*}{\partial r} - \frac{\nu}{r}\psi^* = 0. \tag{3}$$

The central regularity conditions guarantee that there is no singularity of transverse shear stress resultant of the shells when $\beta \geqslant 1$. (The singularity of transverse stress resultant is unavoidable due to material orthotropy when $\beta < 1$ according to theories of orthotropic shells.) The combination of various values of λ and ν leads to four usual boundary conditions:

(1) $\lambda = \nu = \nu_\theta$, corresponding to hinged edge;
(2) $\lambda = \nu_\theta$, $\nu = \infty$, simply-supported edge;
(3) $\lambda = \nu = \infty$, movably clamped edge;
(4) $\lambda = \infty$, $\nu = \nu_\theta$, immovably clamped edge.

The compatibility equation is given as (Dumir, 1986)

$$\mathscr{L}_2(\psi^*) + \frac{h}{2}E_\theta \frac{\partial w^*}{\partial r}\left(\frac{\partial w^*}{\partial r} + 4\frac{H^*}{a^2}r\right) = 0, \tag{4}$$

where $\mathscr{L}_2$ is another partial differential operator

$$\mathscr{L}_2(..) = r\frac{\partial^2(..)}{\partial r^2} + \frac{\partial(..)}{\partial r} - \frac{\beta}{r}(..).$$

To facilitate the subsequent analysis, we introduce the following dimensionless quantities

$$w = \frac{w^*}{h}, \quad \psi = \frac{a\psi^*}{D}, \quad x = \frac{r}{a}, \quad \tau = t\left(\frac{D}{\gamma h a^4}\right)^{1/2}, \quad H = \frac{H^*}{h}, \quad m = \sqrt{\beta}.$$

Using these quantities in eqns (1)–(4) yields nondimensional variational and differential equations with corresponding boundary conditions as

$$\int_{\tau_1}^{\tau_2}\int_0^1 \left\{\frac{\partial^2 w}{\partial \tau^2} + \bar{\mathscr{L}}_1(w) - \frac{1}{x}\frac{\partial}{\partial x}\left[\psi\left(\frac{\partial w}{\partial x} + 2xH\right)\right]\right\}\delta w x \,\mathrm{d}x\,\mathrm{d}\tau = 0, \tag{5}$$

$$\bar{\mathscr{L}}_2(x^m\psi) + 6(\beta - \nu_\theta^2)\frac{\partial w}{\partial x}\left(\frac{\partial w}{\partial x} + 4xH\right) = 0, \tag{6}$$

with

$$x = 0: \quad w \text{ is finite}, \quad \frac{\partial w}{\partial x} = 0, \quad \lim_{x\to 0}\left(x\frac{\partial^3 w}{\partial x^3} + \frac{\partial^2 w}{\partial x^2} - \frac{\beta}{x}\frac{\partial w}{\partial x}\right) = 0, \quad \psi = 0, \tag{7}$$

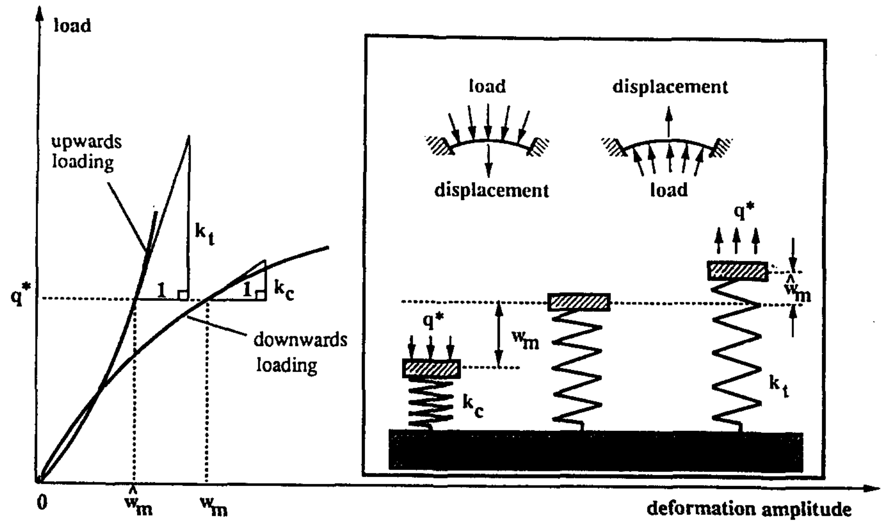

Fig. 2. Illustration of the asymmetrical spring-like behavior of a shallow spherical shell with its simulated nonlinear spring of compressive stiffness k_c and tensile stiffness k_t.

$$x = 1: \quad w = 0, \quad \frac{\partial^2 w}{\partial x^2} + \lambda \frac{\partial w}{\partial x} = 0, \quad \frac{\partial \psi}{\partial x} - \nu\psi = 0. \tag{8}$$

Where $\bar{\mathscr{L}}_1$ and $\bar{\mathscr{L}}_2$ are two nondimensional integrable partial differential operators

$$\bar{\mathscr{L}}_1(..) = \frac{1}{x}\frac{\partial}{\partial x}x^m\frac{\partial}{\partial x}x^{-(2m-1)}\frac{\partial}{\partial x}x^m\frac{\partial}{\partial x}(..), \quad \bar{\mathscr{L}}_2(..) = x^m\frac{\partial}{\partial x}x^{-(2m-1)}\frac{\partial}{\partial x}(..).$$

Before solving eqns (5) and (6), some preliminary physical insight of the present system is explored. By looking at the deflection curves of a shallow spherical shell (Fig. 2), it is observed that the shell exhibits stiffness of different types when being loaded downwards and upwards respectively, much like a mass-spring system with different tensile and compressive stiffnesses varying with deformation magnitudes. Similarities between these two physical systems strongly suggest that some similar dynamic behavior may exist for the two systems. Some important conclusions drawn regarding nonlinear vibrations of a system of single degree-of-freedom (as here is the mass-spring system) focuses our thoughts when putting forward a vibration model of reasonable physical ground. An examination of those closed trajectories around the *static center*, or equilibrium point in the *phase plane* of a system of single degree-of-freedom shows that the trajectories generally do *not* extend the same distance to the right and left of the center (Nayfeh and Mook, 1979), the midpoint of the motion shifts away from the static center as the motion amplitudes increase, the shift is called the *drift* or *steady-streaming* (Fig. 3). This phenomenon of *drift* should be expected to occur in shell vibrations where the structure has asymmetric stiffness. As a result of the above reasoning, the deflection of the shell may be sought in the form:

$$w(x,\tau) = W(x)(\xi + \cos\omega\tau), \tag{9}$$

here ξ is an unspecified parameter modelling the asymmetry (or inequality) of upwards and downwards motion amplitudes of the shell, and can be physically recognized as an intrinsic quantity related to the magnitude of the above-mentioned *drift* (it turns out to be a positive parameter later on). ω is the nondimensional fundamental natural frequency of vibration. Both ξ and ω are quantities related to the geometrical and material parameters, as well as vibration amplitudes and boundary conditions. $W(x)$ is an unspecified spatial function.

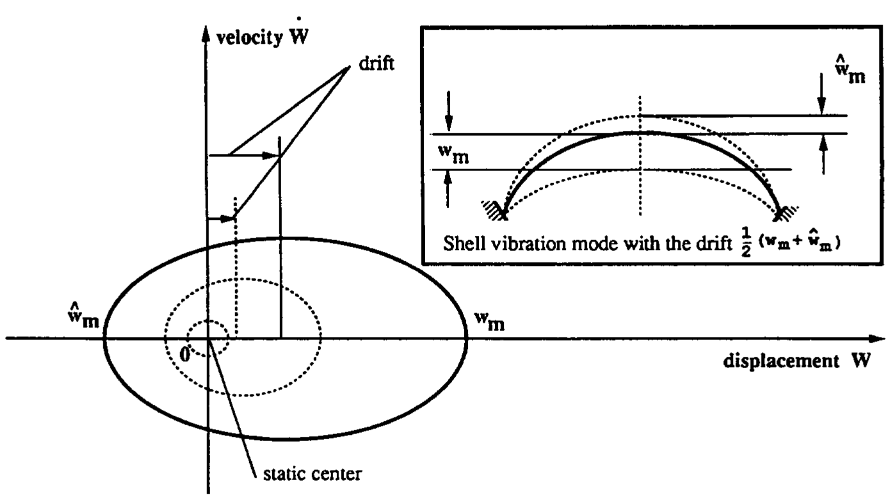

Fig. 3. Illustration of the drift of nonlinear dynamic systems.

From eqn (6), it is further concluded that the stress function should assume the form of

$$\psi(x,\tau) = \Psi(x)(\xi+\cos\omega\tau)+\Phi(x)(\xi+\cos\omega\tau)^2, \tag{10}$$

where both $\Psi(x)$ and $\Phi(x)$ are unspecified spatial functions, the latter clearly represents higher order (nonlinear) vibration effects.

Substituting expressions (9) and (10) into eqn (5), then applying integration with respect to τ over the interval of one vibration period $[0, 2\pi/\omega]$ and using appropriate boundary conditions leads to

$$\int_0^1 \left\{\mathbf{L}(W)-\frac{\omega^2}{(2\xi^2+1)}W-\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left[2xH\Psi+f(\xi)\left(2xH\Phi+\Psi\frac{\mathrm{d}W}{\mathrm{d}x}\right)+g(\xi)\Phi\frac{\mathrm{d}W}{\mathrm{d}x}\right]\right\}\delta Wx\,\mathrm{d}x$$

$$+\frac{1}{(2\xi^2+1)}\left\{\int_0^1\left(\Psi\frac{\mathrm{d}W}{\mathrm{d}x}+2xH\Phi\right)\frac{\mathrm{d}W}{\mathrm{d}x}\mathrm{d}x+\xi\int_0^1\left[2x\mathbf{L}(W)W+4xH\Psi\frac{\mathrm{d}W}{\mathrm{d}x}+3\Phi\left(\frac{\mathrm{d}W}{\mathrm{d}x}\right)^2\right]\mathrm{d}x\right.$$

$$\left.+2\xi^2\int_0^1\left(\Psi\frac{\mathrm{d}W}{\mathrm{d}x}+2xH\Phi\right)\frac{\mathrm{d}W}{\mathrm{d}x}\mathrm{d}x+2\xi^3\int_0^1\Phi\left(\frac{\mathrm{d}W}{\mathrm{d}x}\right)^2\mathrm{d}x\right\}\delta\xi = 0, \tag{11}$$

in which $\mathbf{L}$ is an ordinary differential operator

$$\mathbf{L}(..) = \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}x^m\frac{\mathrm{d}}{\mathrm{d}x}x^{-(2m-1)}\frac{\mathrm{d}}{\mathrm{d}x}x^m\frac{\mathrm{d}}{\mathrm{d}x}(..),$$

$f(\xi)$ and $g(\xi)$ are two rational functions

$$f(\xi) = \frac{2\xi^3+3\xi}{2\xi^2+1},\quad g(\xi) = \frac{8\xi^4+24\xi^2+3}{8\xi^2+4}.$$

Since δW and $\delta\xi$ are arbitrarily assigned function and parameter, the following two equations must hold respectively

$$\mathbf{L}(W)-\frac{\omega^2}{(2\xi^2+1)}W = \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left[2xH\Psi+f(\xi)\left(2xH\Phi+\Psi\frac{\mathrm{d}W}{\mathrm{d}x}\right)+g(\xi)\Phi\frac{\mathrm{d}W}{\mathrm{d}x}\right], \tag{12}$$

with

$$\Lambda_0+\Lambda_1\xi+\Lambda_2\xi^2+\Lambda_3\xi^3 = 0, \tag{13}$$

where the coefficients of the algebraic equation are listed as follows:

$$\Lambda_0 = \int_0^1 \left(\Psi\frac{dW}{dx}+2xH\Phi\right)\frac{dW}{dx}\,dx, \quad \Lambda_2 = 2\Lambda_0,$$

$$\Lambda_3 = 2\int_0^1 \Phi\left(\frac{dW}{dx}\right)^2 dx, \qquad \Lambda_1 = \tfrac{3}{2}\Lambda_3+2\int_0^1\left[x\mathbf{L}(W)W+2xH\Psi\frac{dW}{dx}\right]dx.$$

Placing expressions (9) and (10) into (6)–(8) yields

$$\mathbf{L}^*(x^m\Psi) = 24H(v_\theta^2-\beta)x\frac{dW}{dx}, \tag{14}$$

$$\mathbf{L}^*(x^m\Phi) = 6(v_\theta^2-\beta)\left(\frac{dW}{dx}\right)^2, \tag{15}$$

with boundary conditions for spatial modes

$$x=0:\quad W=\mathbf{W}_0,\quad \frac{dW}{dx}=0,\quad \lim_{x\to 0}\left(x\frac{d^3W}{dx^3}+\frac{d^2W}{dx^2}-\frac{\beta}{x}\frac{dW}{dx}\right)=0,$$

$$\Psi=0,\quad \Phi=0, \tag{16}$$

$$x=1:\quad W=0,\quad \frac{d^2W}{dx^2}+\lambda\frac{dW}{dx}=0,\quad \frac{d\Psi}{dx}-v\Psi=0,\quad \frac{d\Phi}{dx}-v\Phi=0, \tag{17}$$

where $\mathbf{L}^*$ is another ordinary differential operator

$$\mathbf{L}^*(..) = x^m\frac{d}{dx}x^{-(2m-1)}\frac{d}{dx}(..),$$

and $\mathbf{W}_0$ is an unspecified parameter

$$\mathbf{W}_0 = W(x)|_{x=0},$$

which will prove very useful in the next section when we try to obtain a solution of the formulated equations. The relations of $\mathbf{W}_0$ to the nondimensional central inward vibration amplitude W_m (positive value) and outward vibration amplitude $\hat{W}_m$ (negative value) of the shell vibration are easily obtained from modal expression (9) as

$$W_m = \mathbf{W}_0(\xi+1),\quad \hat{W}_m = \mathbf{W}_0(\xi-1). \tag{18}$$

It is obvious that the parameter ξ does indeed represent the extent of the asymmetry of vibration amplitudes of the shells (Fig. 3). The drift of the central vibration amplitude $\mathbf{W}_d$ may be obtained as

$$\mathbf{W}_d = \tfrac{1}{2}(W_m+\hat{W}_m) = \xi\mathbf{W}_0 = \frac{\xi}{\xi+1}W_m.$$

Thus far, the nonlinearly-coupled algebraic-differential eigenvalue equations (12)–(15)

subject to boundary conditions (16), (17) constitute the basic equations describing nonlinear axisymmetrical free vibrations of orthotropic thin shallow spherical shells.

3. ASYMPTOTIC SOLUTIONS

Among all the available asymptotic methods, the *modified iteration method* (Yeh and Liu, 1965) has been well-recognised as a very effective analytic tool handling the static problem of plates and shells (Liu, 1984; Liu and Li, 1988; Li, 1991a). Further, a counterpart of this method dealing with problems of flat plates in dynamic regime was put forward by the author (Li and Liu, 1990, 1991; Li, 1991b–d). Now, a successive iteration method following the essential line of the above-mentioned methods is introduced here to handle eqns (12)–(15): The iteration starts with the *exact* modal solutions of linear (very small amplitude) axisymmetrical vibration of orthotropic thin shallow spherical shells, successive iterations will ensue while the nondimensional factor $\mathbf{W}_0$ functions as an iterative parameter. In the following, all the subscripts assigned to quantities W, Ψ, Φ and ξ indicate the order of iterations.

At first, we put $\xi = 0$ and neglect all nonlinear terms at the right-hand side of eqn (12), thus starting with the first order iteration as

$$\mathbf{L}(W_1) - \omega_L^2 W_1 = \frac{2}{x} H \frac{\mathrm{d}}{\mathrm{d}x}(x\Psi_1), \tag{19}$$

$$\mathbf{L}^*(x^m \Psi_1) = 24H(\nu_\theta^2 - \beta)x \frac{\mathrm{d}W_1}{\mathrm{d}x}, \tag{20}$$

$$\mathbf{L}^*(x^m \Phi_1) = 6(\nu_\theta^2 - \beta)\left(\frac{\mathrm{d}W_1}{\mathrm{d}x}\right)^2, \tag{21}$$

where ω_L is the linear natural frequency of axisymmetric shell vibration. Corresponding boundary conditions are

$$x = 0: \quad W_1 = \mathbf{W}_0, \quad \frac{\mathrm{d}W_1}{\mathrm{d}x} = 0, \quad \lim_{x \to 0}\left(x\frac{\mathrm{d}^3 W_1}{\mathrm{d}x^3} + \frac{\mathrm{d}^2 W_1}{\mathrm{d}x^2} - \frac{\beta}{x}\frac{\mathrm{d}W_1}{\mathrm{d}x}\right) = 0,$$

$$\Psi_1 = 0, \quad \Phi_1 = 0, \tag{22}$$

$$x = 1: \quad W_1 = 0, \quad \frac{\mathrm{d}^2 W_1}{\mathrm{d}x^2} + \lambda\frac{\mathrm{d}W_1}{\mathrm{d}x} = 0, \quad \frac{\mathrm{d}\Psi_1}{\mathrm{d}x} - \nu\Psi_1 = 0, \quad \frac{\mathrm{d}\Phi_1}{\mathrm{d}x} - \nu\Phi_1 = 0. \tag{23}$$

After taking into consideration the central conditions in (22), unified solutions of eqns (19) and (20) are obtained in the form of infinite power series as

$$\Psi_1 = \mathbf{W}_0\left(\zeta \sum_{j=0}^{\infty} A_j x^{4j+m} + \eta \sum_{j=0}^{\infty} B_j x^{4j+m+2} + \sum_{j=0}^{\infty} C_j x^{4j+5}\right), \tag{24}$$

$$W_1 = \mathbf{W}_0\left(\zeta \sum_{j=0}^{\infty} D_j x^{4j+m+3} + \eta \sum_{j=0}^{\infty} E_j x^{4j+m+1} + \sum_{j=0}^{\infty} F_j x^{4j}\right), \tag{25}$$

in which ζ and η are unspecified coefficients, A_j, B_j, C_j, D_j, E_j and F_j are power series coefficients determined by the following recursion formulae:

$$A_0 = B_0 = F_0 = 1, \quad C_0 = \frac{12H(v_\theta^2-\beta)}{(9-m^2)(25-m^2)}\omega_L^2,$$

$$A_{j+1} = \left[\frac{48(v_\theta^2-\beta)H^2(4j+m+1)(4j+m-1)+(4j+2m)4j\omega_L^2}{(4j+2m+4)(4j+2m+2)(4j+m+1)(4j+m-1)(4j+4)(4j+2)}\right]A_j,$$

$$B_{j+1} = \left[\frac{48(v_\theta^2-\beta)H^2(4j+m+3)(4j+m+1)+(4j+2m+2)(4j+2)\omega_L^2}{(4j+2m+6)(4j+2m+4)(4j+m+3)(4j+m+1)(4j+6)(4j+4)}\right]B_j,$$

$$C_{j+1} = \left[\frac{48(v_\theta^2-\beta)H^2(4j+6)(4j+4)+(4j+m+5)(4j-m+5)\omega_L^2}{(4j+m+9)(4j+m+7)(4j-m+9)(4j-m+7)(4j+6)(4j+4)}\right]C_j,$$

$$D_j = \frac{(4j+2m+4)(4j+4)}{24H(v_\theta^2-\beta)(4j+m+3)}A_{j+1}, \quad E_j = \frac{(4j+2m+2)(4j+2)}{24H(v_\theta^2-\beta)(4j+m+1)}B_j,$$

$$F_{j+1} = \frac{(4j+m+5)(4j-m+5)}{24H(v_\theta^2-\beta)(4j+4)}C_j.$$

Obviously, these coefficients can be eventually expressed as *finite* power series in ω_L^2 whose coefficients tend to go to zero with the increase in powers. Placing solutions (24) and (25) into corresponding boundary conditions (23) finally yields a set of linear algebraic equations, from which ω_L, ζ and η are available numerically.

Let us define some quantities that are to be used in the subsequent analysis as follows:

$$A_j^{(1)} = \zeta A_j, \quad B_j^{(1)} = \eta B_j, \quad C_j^{(1)} = C_j, \quad D_j^{(1)} = \zeta D_j, \quad E_j^{(1)} = \eta E_j, \quad F_j^{(1)} = F_j,$$

$$G_j^{(1)} = (4j+m+3)D_j^{(1)}, \quad H_j^{(1)} = (4j+m+1)E_j^{(1)}, \quad I_j^{(1)} = 4(j+1)F_{j+1}^{(1)},$$

$$K_j^{(1)} = \sum_{i=0}^{j} G_i^{(1)}G_{j-i}^{(1)}, \quad L_j^{(1)} = \sum_{i=0}^{j} H_i^{(1)}H_{j-i}^{(1)}, \quad M_j^{(1)} = \sum_{i=0}^{j} I_i^{(1)}I_{j-i}^{(1)},$$

$$N_j^{(1)} = 2\sum_{i=0}^{j} G_i^{(1)}I_{j-i}^{(1)}, \quad O_j^{(1)} = 2\sum_{i=0}^{j} H_i^{(1)}I_{j-i}^{(1)}, \quad P_j^{(1)} = 2\sum_{i=0}^{j} G_i^{(1)}H_{j-i}^{(1)},$$

$$Q_j^{(1)} = \sum_{i=0}^{j} D_i^{(1)}D_{j-i}^{(1)}, \quad R_j^{(1)} = \sum_{i=0}^{j} E_i^{(1)}E_{j-i}^{(1)}, \quad S_j^{(1)} = \sum_{i=0}^{j} F_i^{(1)}F_{j-i}^{(1)},$$

$$T_j^{(1)} = 2\sum_{i=0}^{j} D_i^{(1)}E_{j-i}^{(1)}, \quad U_j^{(1)} = 2\sum_{i=0}^{j} D_i^{(1)}F_{j-i}^{(1)}, \quad V_j^{(1)} = 2\sum_{i=0}^{j} E_i^{(1)}F_{j-i}^{(1)}.$$

Now substituting solution (25) into eqn (21), integrating and using corresponding boundary conditions in (22) and (23), we arrive at

$$\Phi_1 = \mathbf{W}_0^2\left(\tilde{A}x^m + \sum_{j=0}^{\infty} A_j^{(2)}x^{4j+2m+5} + \sum_{j=0}^{\infty} B_j^{(2)}x^{4j+2m+1} + \sum_{j=0}^{\infty} C_j^{(2)}x^{4j+7} + \sum_{j=0}^{\infty} D_j^{(2)}x^{4j+m+6} + \sum_{j=0}^{\infty} E_j^{(2)}x^{4j+m+4} + \sum_{j=0}^{\infty} F_j^{(2)}x^{4j+2m+3}\right), \quad (26)$$

in which

$$A_j^{(2)} = \frac{6(v_\theta^2-\beta)K_j^{(1)}}{(4j+2m+5)^2-\beta}, \quad B_j^{(2)} = \frac{6(v_\theta^2-\beta)L_j^{(1)}}{(4j+2m+1)^2-\beta}, \quad C_j^{(2)} = \frac{6(v_\theta^2-\beta)M_j^{(1)}}{(4j+7)^2-\beta},$$

$$D_j^{(2)} = \frac{6(v_\theta^2-\beta)N_j^{(1)}}{(4j+m+6)^2-\beta}, \quad E_j^{(2)} = \frac{6(v_\theta^2-\beta)O_j^{(1)}}{(4j+m+4)^2-\beta}, \quad F_j^{(2)} = \frac{6(v_\theta^2-\beta)P_j^{(1)}}{(4j+2m+3)^2-\beta}$$

and

$$\tilde{A} = (v-m)^{-1}\Bigg[\sum_{j=0}^{\infty}(4j+2m+5-v)A_j^{(2)} + \sum_{j=0}^{\infty}(4j+2m+1-v)B_j^{(2)} + \sum_{j=0}^{\infty}(4j+7-v)C_j^{(2)} + \sum_{j=0}^{\infty}(4j+m+6-v)D_j^{(2)} + \sum_{j=0}^{\infty}(4j+m+4-v)E_j^{(2)} + \sum_{j=0}^{\infty}(4j+2m+3-v)F_j^{(2)}\Bigg].$$

Further, substituting solutions (24), (25) and (26) into eqn (13) yields the algebraic equation for the asymptotic values of ξ:

$$(\mathbf{W}_0^3\bar{\Lambda}_0)+(3\mathbf{W}_0^4\bar{\Lambda}_3+2\omega_L^2\mathbf{W}_0^2\bar{\Lambda}_1)\xi_1+(2\mathbf{W}_0^3\bar{\Lambda}_0)\xi_1^2+(2\mathbf{W}_0^4\bar{\Lambda}_3)\xi_1^3 = 0, \tag{27}$$

where $\bar{\Lambda}_0$, $\bar{\Lambda}_1$ and $\bar{\Lambda}_3$ are specified coefficients obtained as

$$\bar{\Lambda}_0 = \sum_{j=0}^{\infty}\Bigg\{\frac{2H\sum_{i=0}^{j}G_i^{(1)}A_{j-i}^{(2)}}{4j+3m+9} + \frac{\sum_{i=0}^{j}[B_i^{(1)}K_{j-i}^{(1)}+2H(G_i^{(1)}F_{j-i}^{(2)}+H_i^{(1)}A_{j-i}^{(2)})]}{4j+3m+7}$$

$$+\frac{\sum_{i=0}^{j}[B_i^{(1)}P_{j-i}^{(1)}+A_i^{(1)}K_{j-i}^{(1)}+2H(G_i^{(1)}B_{j-i}^{(2)}+H_i^{(1)}F_{j-i}^{(2)})]}{4j+3m+5}$$

$$+\frac{\sum_{i=0}^{j}(B_i^{(1)}L_{j-i}^{(1)}+A_i^{(1)}P_{j-i}^{(1)}+2HH_i^{(1)}B_{j-i}^{(2)})}{4j+3m+3}$$

$$+\frac{\sum_{i=0}^{j}A_i^{(1)}L_{j-i}^{(1)}}{4j+3m+1} + \frac{\sum_{i=0}^{j}[C_i^{(1)}K_{j-1}^{(1)}+2H(G_i^{(1)}D_{j-i}^{(2)}+I_i^{(1)}A_{j-i}^{(2)})]}{4j+2m+10}$$

$$+\frac{\sum_{i=0}^{j}[B_i^{(1)}N_{j-i}^{(1)}+C_i^{(1)}P_{j-i}^{(1)}+2H(G_i^{(1)}E_{j-i}^{(2)}+H_i^{(1)}D_{j-i}^{(2)}+I_i^{(1)}F_{j-i}^{(2)})]}{4j+2m+8}$$

$$+\frac{\sum_{i=0}^{j}[A_i^{(1)}N_{j-i}^{(1)}+B_i^{(1)}O_{j-i}^{(1)}+C_i^{(1)}L_{j-i}^{(1)}+2H(H_i^{(1)}E_{j-i}^{(2)}+I_i^{(1)}B_{j-i}^{(2)})]}{4j+2m+6}$$

$$+\frac{2H\tilde{A}G_j^{(1)}+\sum_{i=0}^{j}A_i^{(1)}O_{j-i}^{(1)}}{4j+2m+4} + \frac{2H\tilde{A}H_j^{(1)}}{4j+2m+2}$$

$$+\frac{\sum_{i=0}^{j}[C_i^{(1)}N_{j-i}^{(1)}+2H(G_i^{(1)}C_{j-i}^{(2)}+I_i^{(1)}D_{j-i}^{(2)})]}{4j+m+11}$$

$$+\frac{\sum_{i=0}^{j}[B_i^{(1)}M_{j-i}^{(1)}+C_i^{(1)}O_{j-i}^{(1)}+2H(H_i^{(1)}C_{j-i}^{(2)}+I_i^{(1)}E_{j-i}^{(2)})]}{4j+m+9}+\frac{\sum_{i=0}^{j}A_i^{(1)}M_{j-i}^{(1)}}{4j+m+7}$$

$$\left.+\frac{2H\tilde{A}I_j^{(1)}}{4j+m+5}+\frac{\sum_{i=0}^{j}(C_i^{(1)}M_{j-i}^{(1)}+2HI_i^{(1)}C_{j-i}^{(2)})}{4j+12}\right\},$$

$$\bar{\Lambda}_1=\sum_{j=0}^{\infty}\left\{\frac{Q_j^{(1)}}{4j+2m+8}+\frac{T_j^{(1)}}{4j+2m+6}+\frac{R_j^{(1)}}{4j+2m+4}+\frac{U_j^{(1)}}{4j+m+5}+\frac{V_j^{(1)}}{4j+m+3}+\frac{S_j^{(1)}}{4j+2}\right\},$$

$$\bar{\Lambda}_3=\sum_{j=0}^{\infty}\left\{\frac{\sum_{i=0}^{j}K_i^{(1)}A_{j-i}^{(2)}}{4j+4m+10}+\frac{\sum_{i=0}^{j}(K_i^{(1)}F_{j-i}^{(2)}+P_i^{(1)}A_{j-i}^{(2)}}{4j+4m+8}\right.$$

$$+\frac{\sum_{i=0}^{j}(K_i^{(1)}B_{j-i}^{(2)}+L_i^{(1)}A_{j-i}^{(2)}+P_i^{(1)}F_{j-i}^{(2)})}{4j+4m+6}$$

$$+\frac{\sum_{i=0}^{j}(L_i^{(1)}F_{j-i}^{(2)}+P_i^{(1)}B_{j-i}^{(2)})}{4j+4m+4}+\frac{\sum_{i=0}^{j}L_i^{(1)}B_{j-i}^{(2)}}{4j+4m+2}+\frac{\sum_{i=0}^{j}(K_i^{(1)}D_{j-i}^{(2)}+N_i^{(1)}A_{j-i}^{(2)})}{4j+3m+11}$$

$$+\frac{\sum_{i=0}^{j}(K_i^{(1)}E_{j-i}^{(2)}+N_i^{(1)}F_{j-i}^{(2)}+O_i^{(1)}A_{j-i}^{(2)}+P_i^{(1)}D_{j-i}^{(2)})}{4j+3m+9}$$

$$+\frac{\sum_{i=0}^{j}(L_i^{(1)}D_{j-i}^{(2)}+N_i^{(1)}B_{j-i}^{(2)}+O_i^{(1)}F_{j-i}^{(2)}+P_i^{(1)}E_{j-i}^{(2)})}{4j+3m+7}$$

$$+\frac{\tilde{A}K_j^{(1)}+\sum_{i=0}^{j}(L_i^{(1)}E_{j-i}^{(2)}+O_i^{(1)}B_{j-i}^{(2)})}{4j+3m+5}+\frac{\tilde{A}P_j^{(1)}}{4j+3m+3}+\frac{\tilde{A}L_j^{(1)}}{4j+3m+1}$$

$$+\frac{\sum_{i=0}^{j}(K_i^{(1)}C_{j-i}^{(2)}+M_i^{(1)}A_{j-i}^{(2)}+N_i^{(1)}D_{j-i}^{(2)})}{4j+2m+12}$$

$$+\frac{\sum_{i=0}^{j}(M_i^{(1)}F_{j-i}^{(2)}+N_i^{(1)}E_{j-i}^{(2)}+O_i^{(1)}D_{j-i}^{(2)}+P_i^{(1)}C_{j-i}^{(2)})}{4j+2m+10}$$

$$+\frac{\sum_{i=0}^{j}(L_i^{(1)}C_{j-i}^{(2)}+M_i^{(1)}B_{j-i}^{(2)}+O_i^{(1)}E_{j-i}^{(2)})}{4j+2m+8}+\frac{\tilde{A}N_j^{(1)}}{4j+2m+6}+\frac{\tilde{A}O_j^{(1)}}{4j+2m+4}$$

$$+\frac{\sum_{i=0}^{j}(M_i^{(1)}D_{j-i}^{(2)}+N_i^{(1)}C_{j-i}^{(2)})}{4j+m+13}+\frac{\sum_{i=0}^{j}(M_i^{(1)}E_{j-i}^{(2)}+O_i^{(1)}C_{j-i}^{(2)})}{4j+m+11}$$

$$\left.+\frac{\tilde{A}M_j^{(1)}}{4j+m+7}+\frac{\sum_{i=0}^{j}M_i^{(1)}C_{j-i}^{(2)}}{4j+14}\right\}.$$

Thus given values of $\mathbf{W}_0$ we can evaluate ξ_1 numerically using eqn (27). From (18) we then have values of (inward) vibration amplitudes as

$$W_m = \mathbf{W}_0(\xi_1+1). \tag{28}$$

To find the asymptotic values of the nonlinear frequency $\omega_{\rm NL}$, we proceed to a step further and solve the following iterated equations:

$$\mathbf{L}(W_2)-\frac{\omega_{\rm NL}^2}{(2\xi_1^2+1)}W_2=\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left[2xH\Psi_2+f(\xi_1)\left(2xH\Phi_1+\Psi_1\frac{\mathrm{d}W_1}{\mathrm{d}x}\right)+g(\xi_1)\Phi_1\frac{\mathrm{d}W_1}{\mathrm{d}x}\right], \tag{29}$$

$$\mathbf{L}^*(x^m\Psi_2)=24H(v_\theta^2-\beta)x\frac{\mathrm{d}W_2}{\mathrm{d}x}, \tag{30}$$

with corresponding boundary conditions

$$x=0:\quad W_2=\mathbf{W}_0,\quad \frac{\mathrm{d}W_2}{\mathrm{d}x}=0,\quad \lim_{x\to 0}\left(x\frac{\mathrm{d}^3W_2}{\mathrm{d}x^3}+\frac{\mathrm{d}^2W_2}{\mathrm{d}x^2}-\frac{\beta}{x}\frac{\mathrm{d}W_2}{\mathrm{d}x}\right)=0,\quad \Psi_2=0, \tag{31}$$

$$x=1:\quad W_2=0,\quad \frac{\mathrm{d}^2W_2}{\mathrm{d}x^2}+\lambda\frac{\mathrm{d}W_2}{\mathrm{d}x}=0,\quad \frac{\mathrm{d}\Psi_2}{\mathrm{d}x}-v\Psi_2=0. \tag{32}$$

Now with the nonlinear terms being replaced with knowns in terms of the first-order solutions, the above *nonhomogeneous linear* equations are easily integrable. Taking into account the central conditions in (31), we can find the solutions expressed in closed form as

$$\Psi_2=\mathbf{W}_0\left(\hat{\zeta}\sum_{j=0}^{\infty}A_j^{(3)}x^{4j+m}+\hat{\eta}\sum_{j=0}^{\infty}B_j^{(3)}x^{4j+m+2}+\sum_{j=0}^{\infty}C_j^{(3)}x^{4j+5}\right)$$

$$+\mathbf{W}_0^2\left(\sum_{j=0}^{\infty}D_j^{(3)}x^{4j+m+4}+\sum_{j=0}^{\infty}E_j^{(3)}x^{4j+m+6}+\sum_{j=0}^{\infty}F_j^{(3)}x^{4j+2m+3}\right.$$

$$\left.+\sum_{j=0}^{\infty}G_j^{(3)}x^{4j+2m+5}+\sum_{j=0}^{\infty}H_j^{(3)}x^{4j+11}\right)$$

$$+\mathbf{W}_0^3\left(\sum_{j=0}^{\infty}I_j^{(3)}x^{4j+m+6}+\sum_{j=0}^{\infty}J_j^{(3)}x^{4j+m+12}+\sum_{j=0}^{\infty}K_j^{(3)}x^{4j+2m+3}\right.$$

$$\left.+\sum_{j=0}^{\infty}L_j^{(3)}x^{4j+2m+5}+\sum_{j=0}^{\infty}M_j^{(3)}x^{4j+3m+4}+\sum_{j=0}^{\infty}N_j^{(3)}x^{4j+3m+6}+\sum_{j=0}^{\infty}P_j^{(3)}x^{4j+13}\right), \tag{33}$$

$$W_2 = \mathbf{W}_0\left(\hat{\zeta}\sum_{j=0}^{\infty}\bar{A}_j^{(3)}x^{4j+m+3}+\hat{\eta}\sum_{j=0}^{\infty}\bar{B}_j^{(3)}x^{4j+m+1}+\sum_{j=0}^{\infty}\bar{C}_j^{(3)}x^{4j}\right)$$

$$+\mathbf{W}_0^2\left(\sum_{j=0}^{\infty}\bar{D}_j^{(3)}x^{4j+m+3}+\sum_{j=0}^{\infty}\bar{E}_j^{(3)}x^{4j+m+5}+\sum_{j=0}^{\infty}\bar{F}_j^{(3)}x^{4j+2m+2}\right.$$

$$\left.+\sum_{j=0}^{\infty}\bar{G}_j^{(3)}x^{4j+2m+4}+\sum_{j=0}^{\infty}\bar{H}_j^{(3)}x^{4j+10}\right)$$

$$+\mathbf{W}_0^3\left(\sum_{j=0}^{\infty}\bar{I}_j^{(3)}x^{4j+m+5}+\sum_{j=0}^{\infty}\bar{J}_j^{(3)}x^{4j+m+11}+\sum_{j=0}^{\infty}\bar{K}_j^{(3)}x^{4j+2m+2}\right.$$

$$\left.+\sum_{j=0}^{\infty}\bar{L}_j^{(3)}x^{4j+2m+4}+\sum_{j=0}^{\infty}\bar{M}_j^{(3)}x^{4j+3m+3}+\sum_{j=0}^{\infty}\bar{N}_j^{(3)}x^{4j+3m+5}+\sum_{j=0}^{\infty}\bar{P}_j^{(3)}x^{4j+12}\right), \quad (34)$$

where $\hat{\zeta}$ and $\hat{\eta}$ are unspecified coefficients, while $A_j^{(3)}, \bar{A}_j^{(3)}, \ldots, P_j^{(3)}$ and $\bar{P}_j^{(3)}$ can be expressed as *finite* power series in ω_{NL}^2, these series are too lengthy to be listed here.

Substituting solutions (33) and (34) into boundary conditions (32) yields a set of algebraic equations

$$\begin{bmatrix}\alpha_{11} & \alpha_{12} & (\alpha_{13}+\mathbf{W}_0\alpha_{14}+\mathbf{W}_0^2\alpha_{15})\\ \alpha_{21} & \alpha_{22} & (\alpha_{23}+\mathbf{W}_0\alpha_{24}+\mathbf{W}_0^2\alpha_{25})\\ \alpha_{31} & \alpha_{32} & (\alpha_{33}+\mathbf{W}_0\alpha_{34}+\mathbf{W}_0^2\alpha_{35})\end{bmatrix}\begin{bmatrix}\hat{\zeta}\\ \hat{\eta}\\ 1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}, \quad (35)$$

where the matrix elements $\alpha_{ij}(i = 1, 2, 3;\ j = 1, 2, 3, 4, 5)$ can be expressed as *infinite* power series in ω_{NL}^2; they are not listed here for brevity. Now that homogeneous equation (35) permits a *non-trivial* solution as $[\hat{\zeta}, \hat{\eta}, 1]^{\mathrm{T}}$, the determinant of the above matrix must vanish. This condition finally leads to

$$\sum_{j=0}^{\infty}a_j\omega_{\mathrm{NL}}^{2j}+\mathbf{W}_0\left(\sum_{j=0}^{\infty}b_j\omega_{\mathrm{NL}}^{2j}\right)+\mathbf{W}_0^2\left(\sum_{j=0}^{\infty}c_j\omega_{\mathrm{NL}}^{2j}\right)=0. \quad (36)$$

Here coefficients a_j, b_j and c_j are easily determined on computers. In actual computation, all the above series are evaluated with only a few terms to quite satisfactory accuracy, since those series are of fast convergence. Given values of $\mathbf{W}_0$ we can obtain values of W_m and ω_{NL} from eqns (28) and (36) respectively. The amplitude-frequency response relation is finally achieved.

4. NUMERICAL RESULTS AND DISCUSSIONS

Here the case that had been studied by Sinharay *et al.* (1985) is re-examined first. In this case, isotropic thin shallow spherical shells ($\beta = 1$) vibrate with their outer edges being movably or immovably clamped. Numerical computations show that the present results are in agreement with those of Sinharay (Fig. 4). The discrepancy between the two results increase with vibration amplitude W_m for shells with larger values of geometrical parameter H. This may be explained with two points in Sinharay's approach: the neglected coupling term in Sinharay's energy expression for the shells may matter a little more under some circumstances since coupling effects become stronger when the geometrical parameter H and vibration amplitude W_m increase; secondly, Sinharay's amplitude-frequency relation derived from a perturbation method that used W_m as a perturbation parameter only retained the effects of W_m of even power on the frequencies and is only valid for small values of W_m (say, $W_m \leqslant 1$), while here the effects of W_m with odd power are also included in relation

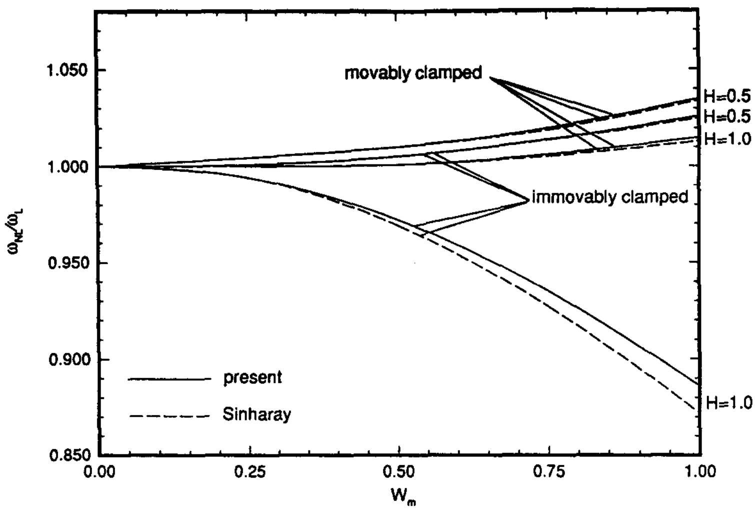

Fig. 4. Comparisons between theoretical results for movably- and immovably-clamped isotropic thin shallow spherical shells ($\beta = 1$, $\nu_\theta = 0.3$).

(36), which is derived by the current method that is valid for a much wider range of W_m (Li, 1991b).

Numerical results in a wider range of vibration amplitudes ($0 \leqslant W_m \leqslant 4.5$) for two additional cases are presented in Figs 5–8, where the shells vibrate with immovably-clamped or hinged edges. It is observed that the geometrical parameter H has great influence on vibration response in that it can change the nonlinearity from hardening type to softening type. When H exceeds certain values for various β, initial softening nonlinearity tends to occur and nonlinearity of hardening type will always be restored with the increase of vibration amplitude W_m. The extent of initial softening increases with H at first, then decreases with the geometrical parameter for the case of hinged edges (Fig. 7). The effect of orthotropy parameter β on vibration is associated with the geometrical parameter H and vibration amplitudes W_m. It is slight for the shells with smaller values of H and smaller vibration amplitudes W_m, and great for the shells with larger values of H and vibration amplitudes W_m. By and large, the nonlinearity and its initial softening, if any, decrease with parameter β. It is also noted that the intrinsic quantity ξ, which represents the extent of

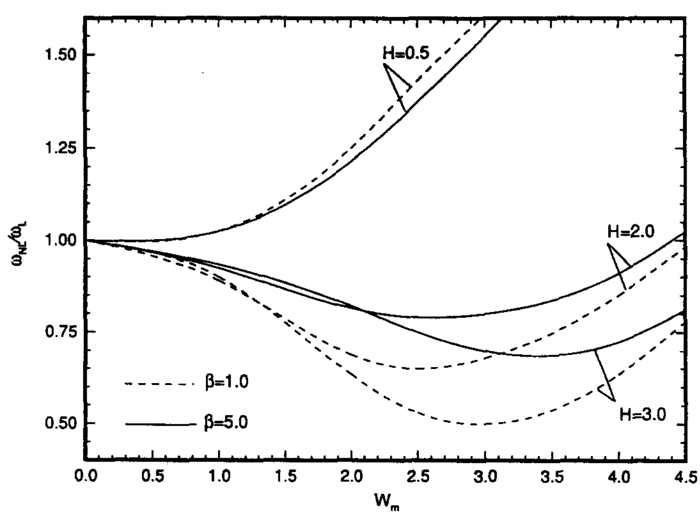

Fig. 5. Effect of the geometrical and orthotropy parameter on vibration frequencies of immovably-clamped shells ($\nu_\theta = 0.3$).

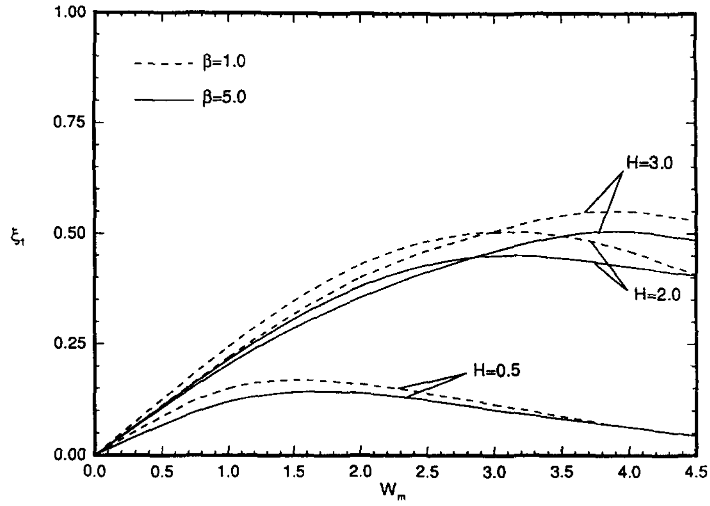

Fig. 6. Effect of the geometrical and orthotropy parameter on the asymmetry of vibration amplitudes of immovably-clamped shells ($\nu_\theta = 0.3$).

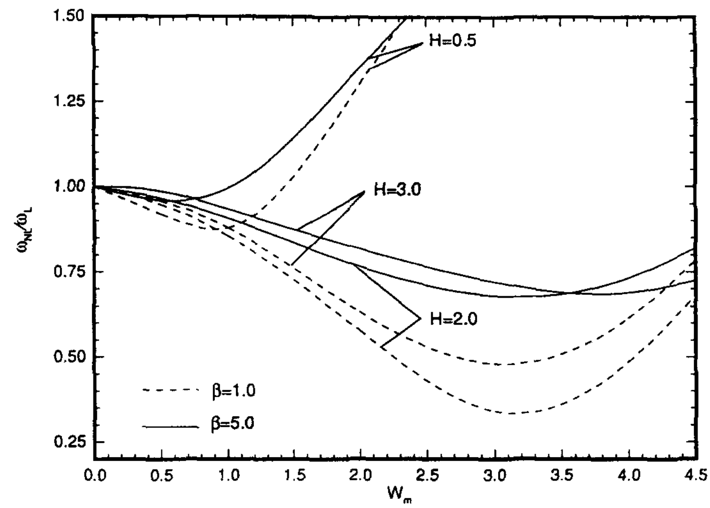

Fig. 7. Effect of the geometrical and orthotropy parameter on vibration frequencies of hinged shells ($\nu_\theta = 0.3$).

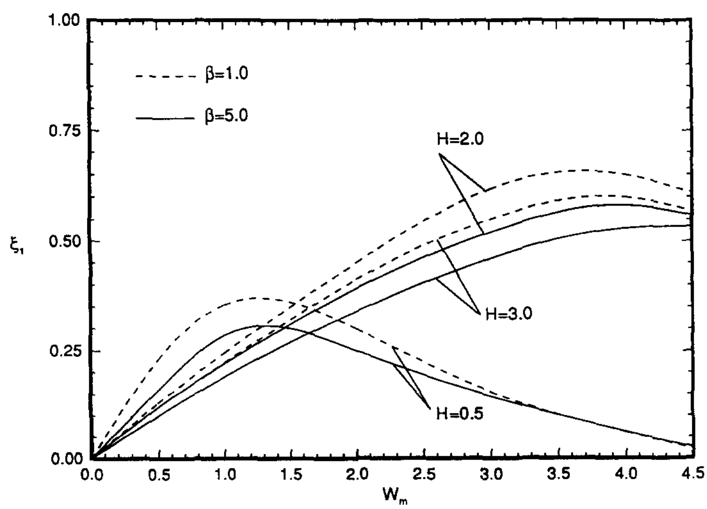

Fig. 8. Effect of the geometrical and orthotropy parameter on the asymmetry of vibration amplitudes of hinged shells ($\nu_\theta = 0.3$).

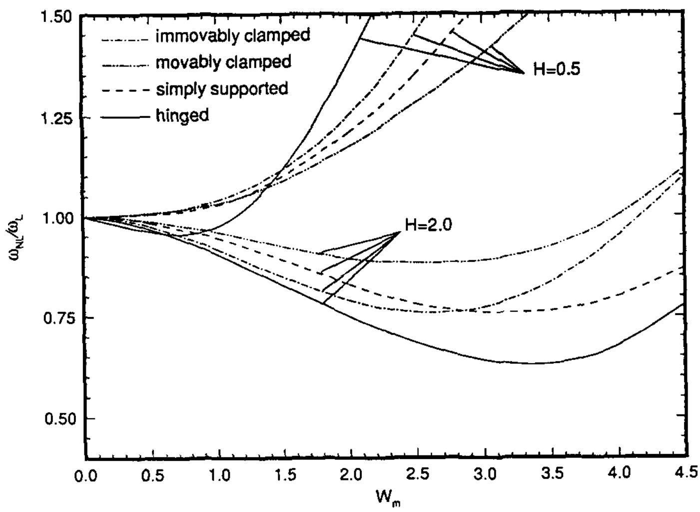

Fig. 9. Effect of various edge supports on vibration frequencies of orthotropic shallow spherical shells ($\nu_0 = 0.3$, $\beta = 3.0$).

asymmetry of vibration amplitudes, increases with H first, but eventually will decrease with H in the case of hinged edges (Fig. 8). Values of ξ will always decrease with β, and increase with W_m at first but subsequently decay after they arrive at their maxima (Figs 6 and 8).

Accordingly, it can be concluded in general, that the nonlinear vibrations of shells with smaller values of H (very shallow shells) are much like those of flat circular plates which exhibit hardening nonlinearity, the extent of the asymmetry of vibration amplitudes is also slight. For shells with larger values of H (moderately shallow shells), the situation is complicated in that both hardening and softening nonlinearities are identified therein: the vibration frequencies decrease with amplitudes first but will eventually increase with amplitudes; the extent of the asymmetry of vibration amplitudes increase with amplitudes initially but will tend to zero with the increase of amplitudes. In one word, the softening of nonlinearities and drift of vibration amplitudes cannot be unbounded, the vibration of shallow spherical shells at larger amplitudes retraces back to what is much like the vibration of flat circular plates: the amplitude-frequency response shows hardening nonlinearity, while the vibration amplitudes become near-symmetrical.

Finally, let us discuss shallow spherical shells subject to four boundary conditions. Results are depicted in Fig. 9. It is observed the nonlinearity of vibration depends on boundary conditions. Vibrations of shells with hinged edge supports demonstrate the strongest nonlinearity and corresponding initial softening. The nonlinearity and its initial softening of vibrations of the shells with movably-clamped edge supports is the weakest. For smaller values of H, vibrations of immovably-clamped shells show stronger non-linearities than those of simply-supported shells. Situations become more complicated when the geometrical parameter H increases where initial softening of nonlinearities varies with various boundary conditions.

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